Some Thoughts on Multislit Emittance Measurements in the Presence of Dispersion and Extraction of the Beam Matrix

Raymond P. Fliller III

March 12, 2008

Abstract

This paper presents some thoughts in the validity of the multislit emittance measurment method in the presence of dispersion and how to extract the three elements of the beam matrix.

1 Introduction

The multislit emittance measurment method has been used at A0 to measure the transverse emittance of the beam. [1] In the past these measurements where performed without any beamline dispersion. With the advent of the Transverse to Longitudinal emittance exchange experiment, the issue can arise where the multislit emittance measurement is performed in a dispersive beamline. In particular, this can occur when the 3.9 GHz cavity is turned off, and the horizontal emittance is measured after the double dogleg.

This measurement consists of two parts. The first is to measure the beam spot size at a given location. The second part is to pass the beam through slits located at the same location as the spot size measurement and to image those beamlets at a downstream location. The geometry is shown in Figure 1. I will not discuss the particular apparatus used, but only describe how the full beam matrix can be measured with this technique, even in the presence of dispersion. I will use results from my Ph.D. thesis [2], in partcular equations 2.66 through 2.74.

2 Some Math

The multislit measurement relies on two parts. The first is the beam spot size at the slits. The second is the angular spread of the beam that exits the slits. To derive these two results, I have more or less cut, paste, and slightly reworded about three pages from my thesis (pgs. 46-49).

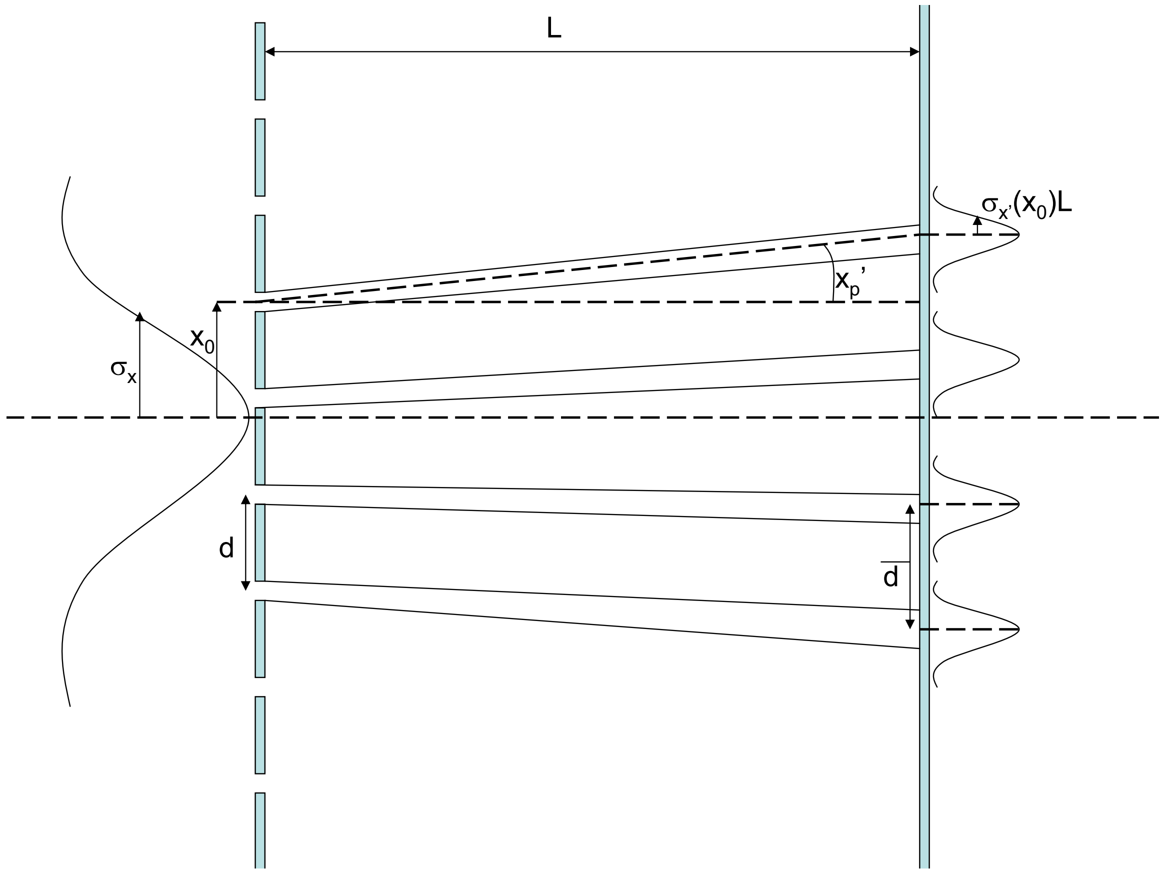

Figure 1: An overhead view of the multislit emittance measurement

The beam distribution in transverse action and longitudinal momentum spread is assumed to be

$$\rho(A, \delta) = \frac{1}{\sqrt{2\pi\sigma_p\epsilon}} \exp\left[-\frac{A}{\epsilon}\right] \exp\left[-\frac{\delta^2}{2{\sigma_p}^2}\right] \tag{1}$$

where A is the invariant of the motion,

$$\begin{aligned} 2A &= \frac{1}{\beta}\left\{(x - D\delta)^2 + [x\alpha + x'\beta - (D\alpha + D'\beta)\delta]^2\right\} & (2) \\ &= \gamma x^2 + 2\alpha x x' + \beta x'^2 \\ &\quad + \left(\gamma D^2 + 2\alpha D D' + \beta D'^2\right)\delta^2 & (3) \\ &\quad -2\left[x(\gamma D + \alpha D') + x'(\alpha D + \beta D')\right]\delta, \end{aligned}$$

ϵ is the RMS unnormalized beam emittance and σ_p is the RMS momentum spread. Because $A = A(x, x', \delta)$ and $\rho(A, \delta) = \rho(x, x', \delta)$, all particle momenta are integrated over to obtain

$\rho(x,x')$. The particle distribution becomes

$$\rho(x,x') = \frac{\exp\left[-\frac{x'^2(\beta\epsilon+D^2\sigma_p{}^2)-2xx'(-\alpha\epsilon+DD'\sigma_p{}^2)+x^2(\gamma\epsilon+D'^2\sigma_p{}^2)}{2\epsilon\sigma_p{}^2\left[\gamma D^2+2\alpha DD'+\beta D'^2+\epsilon/\sigma_p{}^2\right]}\right]}{2\pi\sigma_p\sqrt{\epsilon(\gamma D^2+2\alpha DD'+\beta D'^2+\epsilon/\sigma_p{}^2)}}. \tag{4}$$

It is convenient to calculate the standard deviations of this distribution [3]

$$\sigma_x{}^2 = \epsilon\beta + D^2\sigma_p{}^2 \tag{5a}$$

$$\sigma_{x'}{}^2 = \gamma\epsilon + D'^2\sigma_p{}^2 \tag{5b}$$

$$\sigma_{xx'} = -\alpha\epsilon + DD'\sigma_p{}^2 \tag{5c}$$

which are mean square beam size, mean square divergence and correlation respectively.

The particle angular distribution passing through the slits is given by the conditional probability distribution [3] for a particle to have a position $x_0 \leq x \leq x_0 + \Delta x$ where Δx is the width of the slit and x_0 is the position of the slit

$$\rho(x'|x_0) = \frac{\int\limits_{x_0}^{x_0+\Delta x} \rho(x,x')\mathrm{d}x}{\int\limits_{x_0}^{x_0+\Delta x} \mathrm{d}x \int\limits_{-\infty}^{\infty} \rho(x,x')\mathrm{d}x'}. \tag{6}$$

Using this conditional probability distribution, the average angle and angular spread can be calculated. The average angle of particles passing through the slit is

$$x'_p = \sqrt{\frac{2}{\pi}}\frac{\sigma_{xx'}}{\sigma_x}\frac{\left[1-\exp\left(\frac{\Delta x(2x_0+\Delta x)}{2\sigma_x^2}\right)\right]\exp\left(-\frac{(x_0+\Delta x)^2}{2\sigma_x^2}\right)}{\mathrm{Erf}\left[\frac{x_0}{\sqrt{2}\sigma_x}\right]-\mathrm{Erf}\left[\frac{x_0+\Delta x}{\sqrt{2}\sigma_x}\right]}. \tag{7}$$

The divergence is given by the width of the distribution

$$\begin{aligned}\sigma_{x'}(x_0) = \exp\left[-\frac{(x_0+\Delta x)^2}{2\sigma_x{}^2}\right]\Bigg\{ & \frac{-2}{\pi}\frac{\sigma_{xx'}{}^2}{\sigma_x{}^2}\frac{\left[1-\exp\left(\frac{\Delta x(2x_0+\Delta x)}{2\sigma_x{}^2}\right)\right]^2}{\left[\mathrm{Erf}\left(\frac{x_0}{\sqrt{2}\sigma_x}\right)-\mathrm{Erf}\left(\frac{x_0+\Delta x}{\sqrt{2}\sigma_x}\right)\right]^2} \\ & + \left[\sigma_{x'}^2\exp\left[\frac{(x_0+\Delta x)^2}{2\sigma_x{}^2}\right]+\sqrt{\frac{2}{\pi}}\frac{\sigma_{xx'}^2}{\sigma_x{}^2}\frac{\Delta x+x_0\left[1-\exp\left(\frac{\Delta x(2x_0+\Delta x)}{2\sigma_x{}^2}\right)\right]}{\sigma_x\left[\mathrm{Erf}\left(\frac{x_0}{\sqrt{2}\sigma_x}\right)-\mathrm{Erf}\left(\frac{x_0+\Delta x}{\sqrt{2}\sigma_x}\right)\right]}\right] \\ & \exp\left[\frac{(x_0+\Delta x)^2}{2\sigma_x{}^2}\right]\Bigg\}^{\frac{1}{2}}\end{aligned} \tag{8}$$

Because the slits are very small $\Delta x < \sigma_x$ the difference of the error functions approaches zero, and some of the the exponential terms approach one. So it is convenient to expand Equations 7 and 8 for small Δx.

To lowest order in Δx the average angle of the beam passing through a slit at x_0 is

$$x'_p = \left(x_0 + \frac{\Delta x}{2}\right)\frac{\sigma_{xx'}}{{\sigma_x}^2}. \tag{9}$$

Higher order terms quickly approach zero. In the case of small momentum spread or small normalized dispersion, this is reduced further to

$$x'_p \approx \left(x_0 + \frac{\Delta x}{2}\right)\frac{-\alpha}{\beta} \tag{10}$$

which can be obtained from single particle dynamics for a particle that hits in the middle of the slit.

To fifth order in Δx, the angular spread of the beam passing through the slit becomes

$$\begin{aligned}\sigma_{x'}(x_0) \quad = \quad & \frac{\sqrt{\epsilon\left(\epsilon + \langle\mathcal{H}\rangle{\sigma_p}^2\right)}}{\sigma_x} + \frac{\sigma_{xx'}^2\Delta x^2}{24{\sigma_x}^3\sqrt{\epsilon\left(\epsilon + \langle\mathcal{H}\rangle{\sigma_p}^2\right)}} \\ & \left\{1 - \frac{\Delta x^2}{240{\sigma_x}^2}\left[8 + 5\frac{-\alpha\epsilon\sigma_{xx'} + D^2D'^2{\sigma_p}^4}{\epsilon\left(\epsilon + \langle\mathcal{H}\rangle{\sigma_p}^2\right)} + 12\frac{{x_0}^2}{\sigma_x^2}\right] - \frac{x_0\Delta x^3}{20{\sigma_x}^4}\right\} + O(\Delta x^6)\end{aligned} \tag{11}$$

where $\langle\mathcal{H}\rangle = \gamma D^2 + 2\alpha DD' + \beta D'^2$. For a 50$\mu$m slit, only the first term is important.

The important results of this derivation to determine the emittance are the beam size, Equation 5a, and the divergance through the slit, Equation 11. What is generally done to calculate the emittance is to compute the product of the beam size at the slits (measured on the OTR screen) with the divergance coming through the slits, as measured on a downstream YAG screen.

$$\sigma_x\sigma_{x'}(x_0) = \sqrt{\epsilon\left(\epsilon + \langle\mathcal{H}\rangle{\sigma_p}^2\right)}. \tag{12}$$

In the case of negligible dispersion, this is indeed the unnormalized RMS emittance, otherwise there is an addition term. The question is then how large is this correction to the measurement.

2.1 Numbers

To see what the magnitude of the effect is for the double dogleg (cavity off!), I will use the following numbers: The emittance corresponds to 4 mm-mrad normalized RMS emittance and the momentum spread is measured at 1nC bunch charge chirped at 26 degrees off crest. The beamline parameters are all optimized for this measurement according to Helen's parameters for the transverse and longitudinal phase ellipses.

For these parameters, the measured emittance would be 9e-6 m unnormalized, approximately 63 times larger than the actual unnormalized emittance. By properly tuning the β function, correction can be made smaller. I have made no attempt to minimize the correction. Nonetheless this is still the emittance that one gets from taking the determinant of the transverse beam matrix.

Table 1: Parameters used for Estimation

β	17	m
α	2.11	
γ	0.313	m^{-1}
D	-0.56	m
D'	0.48	
ϵ	1.3e-7	m (RMS unnormalized)
σ_p	1.41%	

3 Measuring the Beam Matrix

Measuring the beam matrix elements from this measurment only relies on further analysis of the same data used to derive the emittance. For the purposes of this section, I will call the result of Equation 12 the emittance, whether of not dispersion is present. The emittance can be defined as the square root of the transverse beam matrix

$$\epsilon^2 = \det \left| \begin{pmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'}^2 \end{pmatrix} \right| \tag{13}$$

The only additional analysis needed is to extract $\sigma_{xx'}$ from the data. Doing this relies on using Equation 9. Please refer to Figure 1 for the remaider of this section. Consider two beamlets a and b on the downstream screen. These slits will be a distance $\bar{d}$ apart given by

$$\bar{d} = x_a + x_a' L - x_b - x_b' L \tag{14}$$

Assume that the slits have a pitch d and beamlet a and b are N beamlets (or equivalently slits) apart, we can write

$$\begin{aligned} \bar{d} &= x_a - x_b + x_a' L - x_b' L \\ &= Nd + (x_a' - x_b')\, L \\ &= Nd + \left(x_a \frac{\sigma_{xx'}}{\sigma_x^2} - x_b \frac{\sigma_{xx'}}{\sigma_x^2} \right) L \\ &= Nd + \frac{\sigma_{xx'}}{\sigma_x^2} NdL \qquad (15) \end{aligned}$$

Where we have assumed that Δx is small compared to the slit size (or equivalently x_i is measured at the slit center). We have also used the result of Equation 9. Subtracting off the distance between the slits on the screen and dividing by the distance between the slits and the downstream screen we arrive at

$$\frac{\sigma_{xx'}}{\sigma_x^2} = \frac{\bar{d} - Nd}{NdL} = \frac{1}{L} \left(\frac{\bar{d}}{Nd} - 1 \right). \tag{16}$$

This gives a determination of $\sigma xx'$ with the beam size measurement. Having the emittance, beam size, and beam correlation allows one to compute the full beam diverence and completes the beam matrix.

In the limit of negligible dispersion, Equation 16 reduces to $\frac{-\alpha}{\beta}$ thereby determining the Courant-Snyder parameters at the location of the slits.

4 Conclusion

I've shown that the multislit method of determining the beam emittance returns a larger emittance than actually exists in the beam in the presence of dispersion. This correction is due to the $\langle \mathcal{H} \rangle$ function and the RMS momentum spread at the location of the slits. Tuning the β function properly can minimize this correction, but some correction of order 1 will always exist.

I have also shown that this method can be used to derive the antire beam matrix at the location of the slits by taking into account the change is the distance of the beamlets as compared to the slits. Special thanks to Tim Koeth for pressing me on the issue of solving this and for transcribing my board notes into some legible form.

References

[1] Yin-e Sun. *Angular-Momentum-Dominated Electron Beams and Flat-Beam Generation.* Ph.D. Thesis. University of Chicago (2005). Rodion Tikhoplav. *Low Emittance Electron Beam Studies.* Ph.D. Thesis University of Rochester (2006).

[2] R. P. Fliller III. *The Crystal Collimation System of the Relativistic Heavy Ion Collider.* Ph.D. Thesis. Stony Brook University (2004).

[3] W. Feller, *An Introduction to Probability Theory and Its Applications. Volume II* (John Wiley & Sons, Inc., second edition, 1971).